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On the Spectral Counting Function for the Dirichlet Laplacian

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We obtain upper and lower bounds for the spectral counting function associated to the Dirichlet laplacian for an open set D in euclidean space, satisfying a geometric and a potential theoretic condition.

1. INTRODUCTION

A theorem of H. Weyl [1] asserts that if Δ_D is the Dirichlet laplacian for an open, bounded set in $\mathbb{R}^m$ with piecewise smooth boundary ∂D, the spectrum of $-\Delta_D$ is discrete: $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots$ and

$$\lim_{\lambda \to \infty} N_D(\lambda)\, \lambda^{-m/2} = (4\pi)^{-m/2}\, (\Gamma(1+m/2))^{-1}\, |D|, \tag{1}$$

where $|D|$ is the volume of D and $N_D(\lambda)$ is the spectral counting function, defined by

$$N_D(\lambda) = \#\{j : \lambda_j < \lambda\}. \tag{2}$$

Weyl's theorem has been generalized to the case where D is an open set in $\mathbb{R}^m$ with finite volume (see, for example, 10.6 in [2]). Various refinements to Weyl's theorem have been made in [3–8] including the case where ∂D is Minkowski measurable [9].

While a necessary and sufficient condition for discreteness of the spectrum of the Dirichlet laplacian for arbitrary open sets has been obtained [10], no general asymptotic formula for the corresponding counting function has been available; detailed asymptotic estimates of the spectral counting function have been obtained in special cases only [11–16].

In this paper we obtain upper and lower bounds for the spectral

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counting function of the Dirichlet laplacian associated to general open sets satisfying the following conditions:

(H_1) There exists a constant $c>0$, such that

$$-\Delta_D \geqslant \frac{c}{d^2(x)} \tag{3}$$

in the sense of quadratic forms, where

$$d(x) = \inf_{y \in \mathbb{R}^m \setminus D} |y-x|. \tag{4}$$

(H_2) For all $\varepsilon>0$, $\mu(\varepsilon)<\infty$, where

$$\mu(\varepsilon) = \int_{\{x \in D : d(x) > \varepsilon\}} dx. \tag{5}$$

The validity of (H_1) has been investigated by several authors [17; 18, 1.5]. We recall the following from [17].

THEOREM 1. *Denote by* Cap(A) *the newtonian capacity of a compact set* $A \subseteq \mathbb{R}^m$ $(m=2, 3, ...)$ *and define for* $x \in \mathbb{R}^m$, $r>0$

$$B(x; r) = \{y \in \mathbb{R}^m : |y-x| \leqslant r\}. \tag{6}$$

Suppose D is an open set in $\mathbb{R}^m$ $(m=2, 3, ...)$ *and suppose there exists a constant* $c_0>0$ *such that*

$$\mathrm{Cap}((\mathbb{R}^m \setminus D) \cap B(x; r)) \geqslant c_0 \, \mathrm{Cap}(B(x; r)), \quad 0<r<\mathrm{diam}(D), x \in \partial D. \tag{7}$$

Then D satisfies (H_1).

Suppose D is an open set in $\mathbb{R}^2$ *and suppose D satisfies* (H_1). *Then D satisfies* (7) *for some constant* $c_0>0$.

Suppose D is an open set in $\mathbb{R}^1$ *with non-empty boundary. Then D satisfies* (H_1) *with* $c=\frac{1}{4}$ *(Hardy's inequality).*

Suppose D is a simply connected open set in $\mathbb{R}^2$ *with non-empty boundary. Then D satisfies* (H_1) *with* $c=\frac{1}{16}$.

Remark 2. Note that Condition (7) is of common use in potential theory. A related condition occurs in a study of the partition function for a bounded open set with a fractal boundary [19].

Remark 3. Note that two-sided bounds on the partition function trace$(e^{t\Delta_D})$ (as obtained for example in 1.9 of [18]) do not imply two-sided bounds on the spectral counting function, unless these bounds give the precise leading term in the asymptotic expansion for $t \downarrow 0$, which in turn should be regularly varying at 0. Note that finiteness of the partition function implies discreteness of the spectrum, while the converse does not hold. We refer to [16], and in particular Corollary 4 in that paper, for related comments and examples.

Remark 4. The conditions (H_1) and (H_2) are independent: the open set $E_1 = \{(x_1, ..., x_m) \in \mathbb{R}^m : x_1 < 0\}$ satisfies (H_1) with $c = \frac{1}{4}$ but does not satisfy (H_2) and the open set $E_2 = \mathbb{R}^2 \setminus (G \times G)$, where $G = \{0, \pm 1, \pm(1+\frac{1}{2}), \pm(1+\frac{1}{2}+\frac{1}{3}), \pm \cdots\}$ satisfies (H_2) with $\mu(\varepsilon) \leqslant 100(\log \varepsilon)^2$ for $\varepsilon > 0$ but does not satisfy (H_1), since $\mathrm{Cap}(\mathbb{R}^2 \setminus E_2) = \mathrm{Cap}(G \times G) = 0$.

The main results of this paper are the following.

THEOREM 5. *Suppose D is an open set in $\mathbb{R}^m$ satisfying* (H_1) *and* (H_2). *Then the spectrum of $-\Delta_D$ is discrete and for all $\lambda > 0$*

$$N_D(\lambda) \leqslant \lambda^{m/2} \mu(c^{1/2}/(2\lambda^{1/2}))\, 5^m \pi^{-m/2} (\Gamma(1+m/2))^{-1}\, (8^{-1} + \pi m c^{-1/2})^m. \tag{8}$$

THEOREM 6. *Suppose D is an open set in $\mathbb{R}^m$ and suppose that the spectrum of $-\Delta_D$ is discrete. Then D satisfies* (H_2). *Furthermore for all $\lambda > 0$*

$$N_D(\lambda) \geqslant \lambda^{m/2} \mu(2\pi m/\lambda^{1/2})\, 4^{-m} \pi^{-m/2} (\Gamma(1+m/2))^{-1}. \tag{9}$$

Theorems 1, 5, and 6 give the following.

COROLLARY 7. *Suppose D is an open set in $\mathbb{R}^m$ satisfying* (H_2). *Suppose that the capacitary density of the boundary ∂D is bounded away from zero (that is,* (7) *holds for some constant $c_0 > 0$). Then there exists a constant $C \in (1, \infty)$ such that for all $\lambda > 0$*

$$\lambda^{m/2} C^{-1} \mu(C \lambda^{-1/2}) \leqslant N_D(\lambda) \leqslant \lambda^{m/2} C \mu(C^{-1} \lambda^{-1/2}). \tag{10}$$

COROLLARY 8. *Suppose D is a simply connected open set in $\mathbb{R}^2$. Then the spectrum of $-\Delta_D$ is discrete if and only if D satisfies* (H_2). *Suppose D satisfies* (H_2). *Then for all $\lambda > 0$*

$$\lambda \mu(4\pi/\lambda^{1/2})(16\pi)^{-1} \leqslant N_D(\lambda) \leqslant \lambda \mu(1/(8\lambda^{1/2}))\, 25\pi^{-1}(8^{-1} + 8\pi)^2. \tag{11}$$

In Section 2 we give the proofs of Theorems 5 and 6. In Section 3 we obtain new results for a concrete example (a spiny urchin).

2. Proofs

Proof of Theorem 5. Let T_ε be the tessellation of $\mathbb{R}^m$ by open hypercubes $Q_1, Q_2, \ldots$ of size ε with vertices at

$$\{(k_1\varepsilon, \ldots, k_m\varepsilon) : k_1 \in \mathbb{Z}, \ldots, k_m \in \mathbb{Z}\}. \tag{12}$$

Define $d_i \colon \mathbb{Z}^+ \to \mathbb{R}$ by

$$d_i = \sup_{x \in Q_i} d(x), \tag{13}$$

and denote by $\Delta^0_{Q_i}$, $(i \in \mathbb{Z}^+)$ the Neumann laplacian on $L^2(Q_i)$. Furthermore define for $i \in \mathbb{Z}^+$ and for $\alpha \in (0, 1)$ the Schrödinger operator

$$H_i^{(\alpha)} = -\alpha\Delta^0_{Q_i} + (1-\alpha)\,\frac{c}{d_i^2}, \tag{14}$$

on $L^2(Q_i)$. By (H_1) and by Neumann bracketing (Proposition 4 on p. 270 in [20])

$$\begin{aligned} -\Delta_D &\geqslant -\alpha\Delta_D + (1-\alpha)\,\frac{c}{d^2(x)} \\ &\geqslant -\alpha\Delta_D + (1-\alpha)\, c \sum_{i=1}^{\infty} \chi_{Q_i}(x)\, d_i^{-2} \\ &\geqslant H_1^{(\alpha)} \oplus H_2^{(\alpha)} \oplus \cdots, \end{aligned} \tag{15}$$

where $\chi_A \colon \mathbb{R}^m \to \{0, 1\}$ is the characteristic function of a set $A \subseteq \mathbb{R}^m$. The spectrum of $H_i^{(\alpha)}$ is given by

$$\left\{\alpha \sum_{j=1}^{m} (\pi k_j/\varepsilon)^2 + (1-\alpha)\, c/d_i^2 : (k_1, \ldots, k_m) \in (\mathbb{Z}^+ \cup \{0\})^m\right\}. \tag{16}$$

Let $N(H_i^{(\alpha)}; \lambda)$ be the counting function associated to $H_i^{(\alpha)}$. Then we have by (15)

$$N_D(\lambda) \leqslant \sum_{i \geqslant 1} N(H_i^{(\alpha)}; \lambda). \tag{17}$$

Define

$$\varepsilon_0 = ((1-\alpha)\, c/\lambda)^{1/2}, \tag{18}$$

and define for $\varepsilon > 0$

$$D_\varepsilon = \{x \in D : d(x) > \varepsilon\}. \tag{19}$$

Then $Q_i \subseteq \mathbb{R}^m \backslash D_{\varepsilon_0}$ implies $N(H_i^{(\alpha)}; \lambda) = 0$. Moreover

$$N(H_i^{(\alpha)}; \lambda) \leqslant \# \left\{ (k_1, ..., k_m) \in (\mathbb{Z}^+ \cup \{0\})^m : \alpha \sum_{j=1}^{m} (\pi k_j/\varepsilon)^2 < \lambda \right\}$$
$$\leqslant 2^{-m} \pi^{m/2} (\Gamma(1+m/2))^{-1} (\varepsilon \lambda^{1/2}/(\pi \alpha^{1/2}) + m^{1/2})^m. \tag{20}$$

Hence (17) becomes

$$N_D(\lambda) \leqslant \lambda^{m/2} (4\pi)^{-m/2} (\Gamma(1+m/2))^{-1} (\varepsilon \alpha^{-1/2} + \pi (m/\lambda)^{1/2})^m$$
$$\cdot \# \{ i : Q_i \cap D_{\varepsilon_0} \neq \phi \}. \tag{21}$$

Since

$$\operatorname{diam}(Q_i) = \varepsilon m^{1/2}, \tag{22}$$

we have for $\varepsilon m^{1/2} < \varepsilon_0$

$$\# \{ i : Q_i \cap D_{\varepsilon_0} \neq \phi \} = \varepsilon^{-m} \sum_{\{i : Q_i \cap D_{\varepsilon_0} \neq \phi\}} \operatorname{meas}(Q_i)$$
$$\leqslant \varepsilon^{-m} \operatorname{meas} \left(\bigcup_{\{i : Q_i \cap D_{\varepsilon_0} \neq \phi\}} Q_i \right)$$
$$\leqslant \varepsilon^{-m} \operatorname{meas}(D_{\varepsilon_0 - \varepsilon\sqrt{m}}) = \varepsilon^{-m} \mu(\varepsilon_0 - \varepsilon m^{1/2}). \tag{23}$$

From (21) and (23) we obtain

$$N_D(\lambda) \leqslant \lambda^{m/2} \mu(\varepsilon_0 - \varepsilon m^{1/2}) (4\pi)^{-m/2} (\Gamma(1+m/2))^{-1} (\alpha^{-1/2} + \pi m^{1/2}/(\varepsilon \lambda^{1/2}))^m. \tag{24}$$

Choose

$$\alpha = \tfrac{16}{25}, \tag{25}$$

and

$$\varepsilon = 10^{-1} (c/(m\lambda))^{1/2}, \tag{26}$$

and Theorem 5 follows from (18), (24), (25), and (26).

Proof of Theorem 6. Let T_ε be the tessellation as in the proof of Theorem 5. Then by Dirichlet bracketing (Proposition 4 on p. 270 of [20])

$$-\Delta_D \leqslant \bigoplus_{\{i : Q_i \cap D_{\varepsilon\sqrt{m}} \neq \phi\}} -\Delta_{Q_i}. \tag{27}$$

The spectrum of $-\Delta_{Q_i}$ is given by

$$\left\{\sum_{j=1}^{m} (\pi k_j/\varepsilon)^2 : (k_1, ..., k_m) \in (\mathbb{Z}^+)^m\right\}. \tag{28}$$

Hence for $\lambda > m(\pi/\varepsilon)^2$

$$N_{Q_i}(\lambda) \geqslant 2^{-m}\pi^{m/2}(\Gamma(1+m/2))^{-1}(\varepsilon\lambda^{1/2}\pi^{-1} - m^{1/2})^m, \tag{29}$$

so that from (27) and (29) for $\lambda > m(\pi/\varepsilon)^2$

$$N_D(\lambda) \geqslant 2^{-m}\pi^{m/2}(\Gamma(1+m/2))^{-1}(\varepsilon\lambda^{1/2}\pi^{-1} - m^{1/2})^m \;\#\{i : Q_i \cap D_{\varepsilon\sqrt{m}} \neq \phi\}. \tag{30}$$

But

$$\begin{aligned}\#\{i : Q_i \cap D_{\varepsilon\sqrt{m}} \neq \phi\} &= \varepsilon^{-m} \sum_{\{i : Q_i \cap D_{\varepsilon\sqrt{m}} \neq \phi\}} \operatorname{meas}(\bar{Q}_i) \\ &= \varepsilon^{-m} \operatorname{meas}\Bigg(\bigcup_{\{i : Q_i \cap D_{\varepsilon\sqrt{m}} \neq \phi\}} \bar{Q}_i\Bigg) \\ &\geqslant \varepsilon^{-m} \operatorname{meas}(D_{\varepsilon\sqrt{m}}) = \varepsilon^{-m}\mu(\varepsilon m^{1/2}). \end{aligned} \tag{31}$$

Choose

$$\varepsilon = 2\pi(m/\lambda)^{1/2}, \tag{32}$$

and Theorem 6 follows from (30), (31), and (32).

3. Spectrum of the Dirichlet Laplacian on a Spiny Urchin

In this section we obtain bounds for the spectral counting function for the Dirichlet laplacian on a spiny urchin.

Definition 9. A set U in $\mathbb{R}^2$ is a spiny urchin if $U = \mathbb{R}^2 \backslash \partial U$, where (in polar coordinates)

$$\partial U = \bigcup_{k=1}^{\infty} \{(r, \theta) : r \geqslant a_k,\ \theta = \pi n 2^{-k},\ n = 1, 2, 3, ..., 2^{k+1}\}, \tag{33}$$

for some sequence $a_1 < a_2 < a_3 < \cdots$ with $a_1 > 0$ and $\lim_{k \to \infty} a_k = \infty$.

The urchin U_1 corresponding to $a_k = k$ was considered on p. 151 in [21]. There it was shown that the embedding of the Sobolev space $W_0^{1,2}(U_1)$

into $L^2(U_1)$ is compact, and hence that $-\Delta_{U_1}$ has discrete spectrum. Moreover, by a result on p. 152 in [12]

$$N_{U_1}(\lambda) \sim \lambda(\log \lambda)^2, \tag{34}$$

($A(\lambda) \sim B(\lambda)$ if there exists $C \in (1, \infty)$ such that $C^{-1}B(\lambda) \leqslant A(\lambda) \leqslant CB(\lambda)$ for $\lambda \geqslant C$).

The result for the general case is the following.

THEOREM 10. *Let U be a spiny urchin. Then the spectrum of $-\Delta_U$ is discrete if and only if*

$$\lim_{k \to \infty} a_k 2^{-k} = 0. \tag{35}$$

Suppose U satisfies (35). *Then for all*

$$\lambda \geqslant 2^{14} a_1^{-2}, \tag{36}$$

$$(32)^{-1} \lambda a_{l(\lambda)}^2 \leqslant N_U(\lambda) \leqslant 50(8^{-1} + 8\pi)^2 \lambda a_{k(\lambda)}^2, \tag{37}$$

where

$$k(\lambda) = \max\{k \in \mathbb{Z}^+ : a_k 2^{-k} > (32)^{-1} \lambda^{-1/2}\} \tag{38}$$

and

$$l(\lambda) = \max\{k \in \mathbb{Z}^+ : a_k 2^{-k} > 64 \lambda^{-1/2}\}. \tag{39}$$

Proof. U is open and simply connected in $\mathbb{R}^2$. By Corollary 8 it is sufficient to show that (H_2) is equivalent to (35). First we obtain a lower bound for $\mu(\varepsilon)$. Let $N(r)$ be the number of endpoints of spines of ∂U in the closed ball $\{x \in \mathbb{R}^2 : |x| \leqslant r\}$. Then for any $r > 0$

$$\mu(\varepsilon) > \pi r^2 - 2^{-1}\pi\varepsilon^2 N(r+\varepsilon) - 2\varepsilon \left\{4(r-a_1)^+ + \sum_{k \geqslant 2} 2^k (r-a_k)^+\right\}, \tag{40}$$

where $(p)^+ = (p + |p|)/2$ for $p \in \mathbb{R}$. Hence

$$\begin{aligned} \mu(\varepsilon) &\geqslant \pi r^2 - 2^{-1}\pi\varepsilon^2 N(r+\varepsilon) - 2\varepsilon r N(r) \\ &\geqslant \pi r^2 - (2^{-1}\pi\varepsilon^2 + 2\varepsilon r)\, N(r+\varepsilon). \end{aligned} \tag{41}$$

Let $\varepsilon < a_1$ and put

$$r = a_k - \varepsilon. \tag{42}$$

Then for $k \in \mathbb{Z}^+$ we have by (41), (42)

$$\mu(\varepsilon) \geqslant \pi(a_k - \varepsilon)^2 - (2^{-1}\pi\varepsilon^2 + 2\varepsilon(a_k - \varepsilon))\, 2^{k+1}$$
$$\geqslant \pi a_k^2 - \varepsilon a_k 2^{k+3}. \tag{43}$$

Suppose (35) is not satisfied. Then there exist a $\delta > 0$ and a subsequence $k_1, k_2, \ldots \in \mathbb{Z}^+$ such that

$$a_{k_i} 2^{-k_i} \geqslant \delta, \qquad i \in \mathbb{Z}^+. \tag{44}$$

Then $\mu(\varepsilon) = +\infty$ for $\varepsilon < \min\{\pi\delta/8, a_1\}$ by (43), (44), and (H_2) does not hold. Suppose (35) is satisfied. Define for $k \geqslant 2$

$$\delta_k = a_k \tan(\pi 2^{-k}). \tag{45}$$

Consider the closed line segments with length $2\delta_k$ whose midpoints are at the endpoints of the spines at distance a_k from the origin and orthogonal to the spines. Let

$$n(\varepsilon) = \max\{n \in \mathbb{Z}^+ : \delta_n > \varepsilon\}. \tag{46}$$

Then $n(\varepsilon) \geqslant 2$ for $\varepsilon < a_1$ and $n(\varepsilon) < \infty$ by (35). For $\varepsilon < a_1$

$$\mu(\varepsilon) \leqslant \pi\{a_{n(\varepsilon)}/\cos(\pi 2^{-n(\varepsilon)})\}^2 \leqslant 2\pi(a_{n(\varepsilon)})^2, \tag{47}$$

so that (H_2) does hold. Hence (35) is equivalent to (H_2).

To prove the upper bound in (37) we note that by (46) and (38)

$$n(8^{-1}\lambda^{-1/2}) = \max\{n \in \mathbb{Z}^+ : a_n \tan(\pi 2^{-n}) > 8^{-1}\lambda^{-1/2}\} \leqslant k(\lambda). \tag{48}$$

By Corollary 8, (47), and (48)

$$N_U(\lambda) \leqslant \lambda\mu(8^{-1}\lambda^{-1/2})\, 25\pi^{-1}(8^{-1} + 8\pi)^2$$
$$\leqslant 50(8^{-1} + 8\pi)^2\, \lambda a_{k(\lambda)}^2, \tag{49}$$

for $8^{-1}\lambda^{-1/2} < a_1$. To prove the lower bound in (37) we note that $l(\lambda) \geqslant 1$ by (36), and

$$N_U(\lambda) \geqslant \lambda\mu(4\pi\lambda^{-1/2})(16\pi)^{-1}$$
$$\geqslant \lambda\{\pi a_{l(\lambda)}^2 - \varepsilon a_{l(\lambda)} 2^{l(\lambda)+3}\}(16\pi)^{-1}$$
$$\geqslant (32)^{-1}\, \lambda a_{l(\lambda)}^2, \tag{50}$$

by Corollary 8, (39), and (43).

EXAMPLE 12. Let U be a spiny urchin with

$$a_k = k^x, \tag{51}$$

for some $x > 0$. Then

$$N_U(\lambda) \sim \lambda(\log \lambda)^{2x}. \tag{52}$$

EXAMPLE 13. Let U be a spiny urchin with

$$a_k = x^k, \tag{53}$$

for some $x \in (1, 2)$. Then

$$N_U(\lambda) \sim \lambda^{\log 2/(\log 2 - \log x)}. \tag{54}$$

EXAMPLE 14. Let U be a spiny urchin with

$$a_k = 2^k k^{-\alpha}, \tag{55}$$

for some $\alpha > 0$. Then

$$\log N_U(\lambda) \sim \lambda^{1/(2\alpha)}. \tag{56}$$

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